MULTIVARIATE MODELING OF LOW, MODERATE, AND LARGE POSITIVE VALUES WITHOUT THRESHOLD SELECTION STEPS

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Cornell University

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Statistics > Methodology

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Multivariate distributional modeling of low, moderate, and large intensities without threshold selection steps

Carlo Gaetan, Philippe Naveau

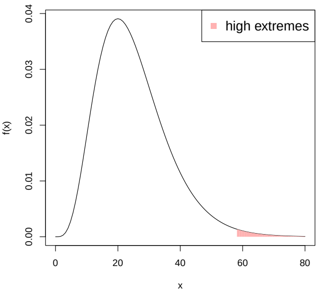
Warm up

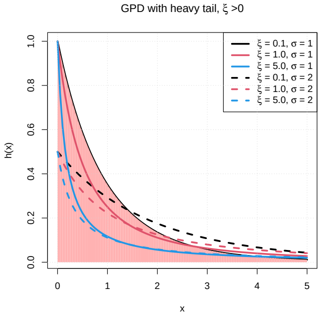
GENERALIZED PARETO DISTRIBUTION (GPD)

The distribution of X, when X exceeds a high threshold u, can be approximated by a GPD

$$H_\xi((x-u)/\sigma) = \begin{cases} 1-(1+\xi(x-u)/\sigma)_+^{-1/\xi} & \text{for } \xi \neq 0 \\ 1-\exp(-(x-u)/\sigma) & \text{for } \xi = 0 \end{cases}$$

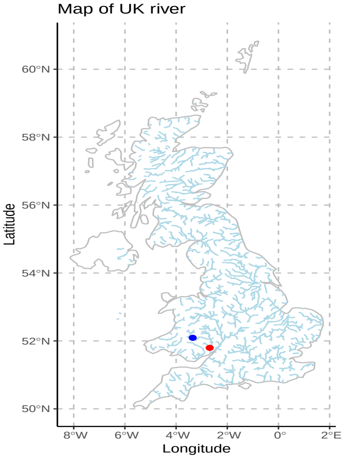
ξ **shape parameter**, $\sigma > 0$ **scale parameter** and $a_+ = \max(a, 0)$.

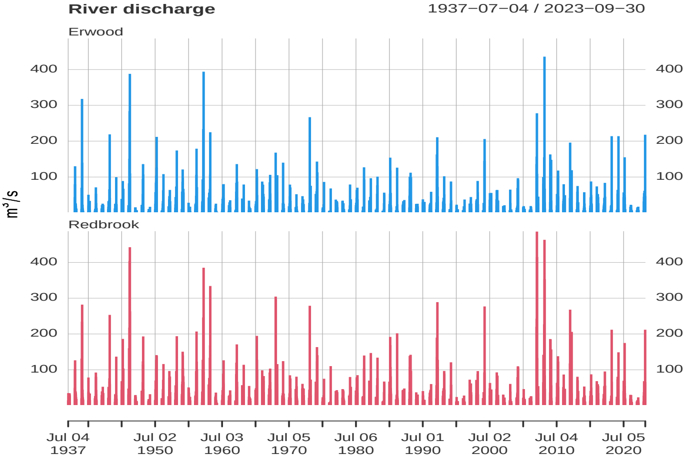




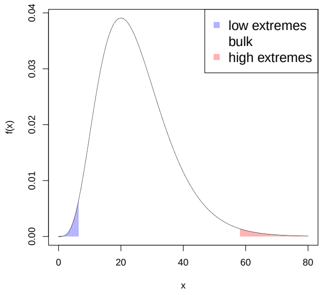
Example

EXAMPLE: WEEKLY MAXIMUM SUMMER RIVER DISCHARGES OF WYE RIVER





RIVER DISCHARGES



- **Flood risk managers** often focus on the analysis of high river flows
- **Farmers** may be interested in periods of low river runoffs to prevent food production shortages
- **Energy producers** in charge of electrical dams can be concerned by the full range of the variable of interest

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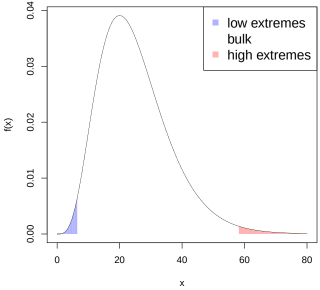

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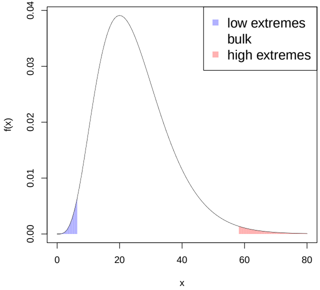

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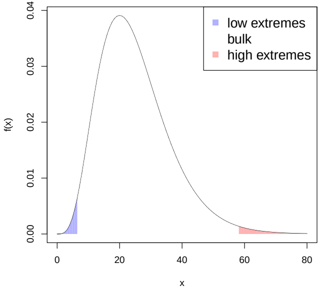

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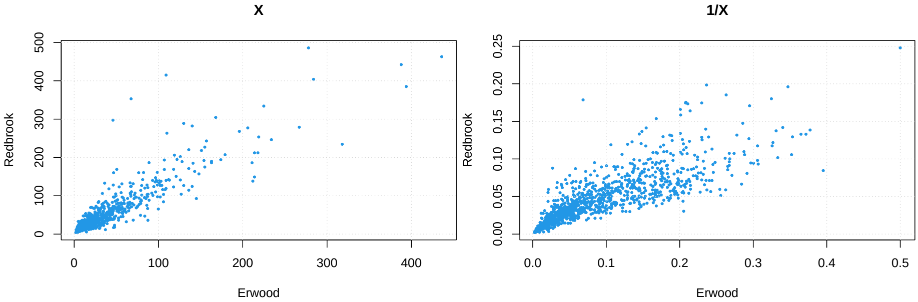
Sites along the same river basin as nearby measurements can be strongly dependent



What is an Extended Generalized Pareto Distribution ?

Water Resources Research

Research Article | Free Access

Modeling jointly low, moderate, and heavy rainfall intensities without a threshold selection

Philippe Naveau, Raphael Huser, Pierre Ribereau, Alexis Hannart

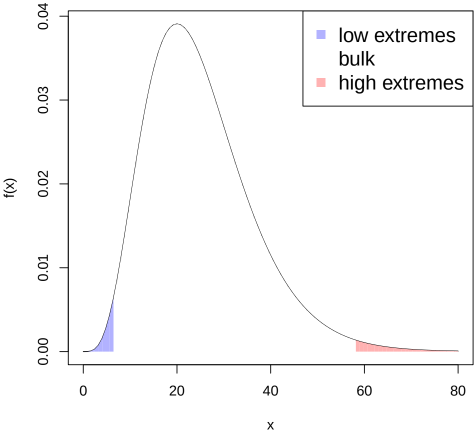
First published: 04 March 2016 | https://doi.org/10.1002/2015WR018552 | Citations: 110

Volume 52, Issue 4
April 2016
Pages 2753-2769

Figures | References | Related | Information

- Distribution for positive data !

TWO STRATEGIES TO MODEL JOINTLY EXTREMES AND BULK



Classical approach

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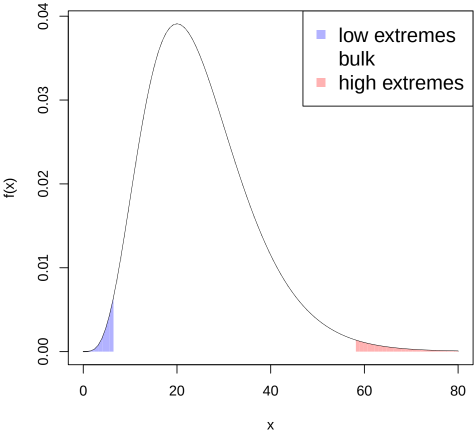

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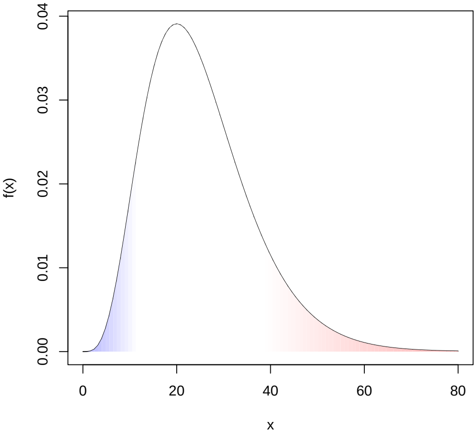

EGPD approach

THE THREE INGREDIENTS FOR A EGPD

High extremes

ξ the GPD parameter of X

THE THREE INGREDIENTS FOR A EGPD

Low extremes

$1/\kappa$ the GPD parameter of $1/X$

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$B(u)$ a CDF on $[0, 1]$ with a pdf $b(u)$

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$$\Pr(X \leq x) = B(H_{\xi}(x)^{\kappa})$$

where the pdf $b(u)$ is such that

$$0 < b(0) < \infty \qquad \text{and} \qquad 0 < b(1) < \infty$$

Low extremes

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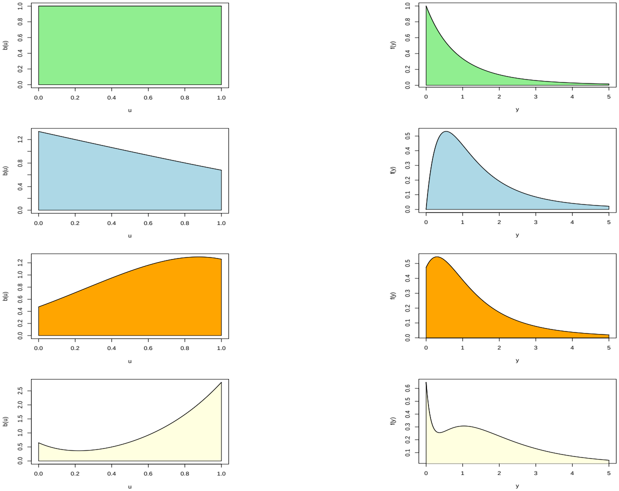
- Scale parameter σ is absorbed in $B(u)$

EGPD EXAMPLES

$b(u)$ EGPD density



TAKE HOME MESSAGE

- $X \sim \text{EGPD}(\kappa, \xi, B)$
- $1/X \sim \text{EGPD}(1/\xi, 1/\kappa, \tilde{B})$
- $B(\cdot) \rightarrow \tilde{B}(\cdot)$

The inverse $1/X$ of a EGPD r.v. is still an EGPD

LEMMA

If

$$X \sim EGPD(\kappa, \xi, B), \quad \Leftrightarrow \quad \Pr(X \leq x) = B(H_{\xi}(x)^{\kappa})$$

then

$$1/X \sim \mathrm{EGPD}(1/\xi, 1/\kappa, \tilde{B}), \quad \Leftrightarrow \quad \Pr(1/X \leq \tilde{x}) = \tilde{B}(H_{1/\kappa}(\tilde{x})^{1/\xi})$$

with

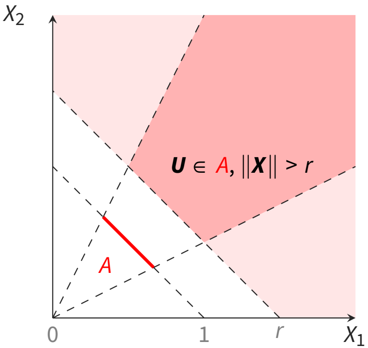
$$\tilde{b}(0) = \kappa\, \xi^{1/\xi}\, b(1) \quad \text{and} \quad \tilde{b}(1) = \xi\, b(0)$$

A multivariate EGPD

POLAR EXTREMES COORDINATES OF $\boldsymbol{X} = (X_1, X_2)$

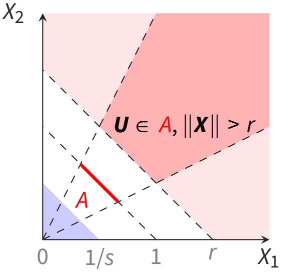
$$\boldsymbol{X} = \|\boldsymbol{X}\| \times \boldsymbol{U}$$

pseudo-radius $\|\boldsymbol{X}\| = X_1 + X_2$ and pseudo-angle $\boldsymbol{U} = (X_1/\|\boldsymbol{X}\|, X_2/\|\boldsymbol{X}\|)$



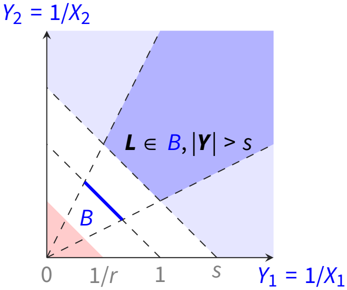
$$\boldsymbol{X} = \|\boldsymbol{X}\| \times \boldsymbol{U} \qquad \boldsymbol{Y} = |\boldsymbol{Y}| \times \boldsymbol{L}$$

Upper extreme's representation



Radius $\|\boldsymbol{X}\| = X_1 + X_2$ and $\boldsymbol{U} = \frac{\boldsymbol{X}}{\|\boldsymbol{X}\|}$

Lower extreme's representation



$|\boldsymbol{Y}| = \frac{1}{\|\boldsymbol{1}/\boldsymbol{Y}\|}$ and $\boldsymbol{L} = \frac{\boldsymbol{Y}}{|\boldsymbol{Y}|}$

TAKE HOME MESSAGE

- $\boldsymbol{X} = (X_1, \ldots, X_d)$ with $X_i \sim \text{EGPD}(\kappa, \xi, B)$ for all $i = 1, \ldots, d$.
- Distribution of $\|\boldsymbol{X}\| = X_1 + \cdots + X_d$

The sum of dependent EGPDs is still an EGPD

- The upper tail parameter, ξ, remains unchanged from marginal behaviors to $\|\boldsymbol{X}\|$, while the lower tail behavior can be have a changing κ.
- $B(\cdot) \longrightarrow B_d(\cdot)$

PROPOSITION: DISTRIBUTION OF $\|\boldsymbol{X}\| = X_1 + \cdots + X_d$

Let be $\boldsymbol{X} = (X_1, \ldots, X_d)$ with $X_i \sim \text{EGPD}(\kappa, \xi, B)$ for all $i = 1, \ldots, d$.
If there exist three positive and finite constants, a, c_- and c_+ such that

$$\lim_{x \to \infty} \frac{\Pr(\|\mathbf{X}\| > x)}{\Pr(X_i > x)} = c_+ \quad (1)$$

and

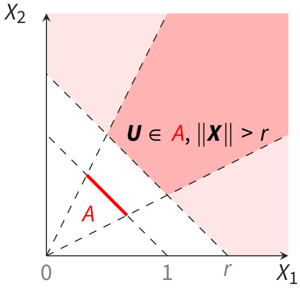
$$\lim_{x \to 0^+} \frac{\Pr(\|\mathbf{X}\| \leq x)}{[\Pr(X_i \leq x)]^a} = c_-, \quad (2)$$

then there exists a CDF B_d such that $\|\mathbf{X}\| \sim \text{EGPD}(a\kappa, \xi, B_d)$ with

$$b_d(0) = c_- \, b(0)^a \text{ and } b_d(1) = c_+ \, b(1)/a.$$

- Moreover $|\boldsymbol{Y}| = 1/\|\boldsymbol{X}\|$ is also $\text{EGPD}(1/\xi, 1/(a\kappa), \tilde{B}_d)$

MULTIVARIATE REGULAR VARIATION DISTRIBUTION



$$\mathbf{X} = \|\boldsymbol{X}\| \times \boldsymbol{U}$$

- $\|\mathbf{X}\|$ independent of $\boldsymbol{U}$ when $\|\mathbf{X}\|$ gets large
- $\Pr(\boldsymbol{U} \in A \mid \|\mathbf{X}\| > r)$ has a non-degenerate limit as $r \to \infty$, i.e.

$$\lim_{r \to \infty} \Pr(\boldsymbol{U} \in A \mid \|\mathbf{X}\| > r) = \Pr(\boldsymbol{U} \in A)$$

The main differences with classical EVT modelling are that

1. Our interest is not only on the upper extremal behaviour of $\boldsymbol{X}$, but also its lower extremal behaviour
2. The radial component is assumed to follow a EGPD, and consequently be in compliance with EVT for both small and large values of $||\mathbf{X}||$
3. In contrast to classical regular variation principles, the radial component **is not necessarily assumed independent** of the angular component
4. In particular, the degree of dependence will change according to the value $||\mathbf{X}||$

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BIVARIATE EGPD $\boldsymbol{X} = (X_1, X_2)$, WITH FOUR INGREDIENTS

$$X_1 = \|\boldsymbol{X}\| U_1 \quad \text{and} \quad X_2 = \|\boldsymbol{X}\| U_2 = \|\boldsymbol{X}\|(1 - U_1)$$

$$\|\boldsymbol{X}\| \sim EGPD(\kappa, \xi, B_d)$$

Low extremes

κ_d the GPD parameter of $1/\|\boldsymbol{X}\|$

Bulk

B_d a CDF function on $[0, 1]$ (with PDF b_d)

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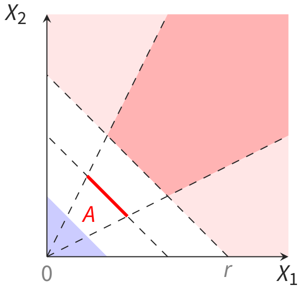

$\boldsymbol{U} = \boldsymbol{X}/\|\boldsymbol{X}\|$

Bivariate conditional model

$$\left[\log\left(\frac{U_1}{1 - U_1}\right) \middle| \|\boldsymbol{X}\| = r\right] \stackrel{d}{=} \delta(r) Z$$

with Z standardized Gaussian $\perp \|\boldsymbol{X}\|$

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- If $\delta(r)$ remains constant for large values of r, then we are in the multivariate regular variation framework
- We can specify other conditional distribution

$$U_1 \,|\, \|\boldsymbol{X}\| = r \sim f(\cdot; \delta(r))$$

  under the constraint $\mathbb{E}\left(U_1 \,|\, \|\boldsymbol{X}\| = r\right) = 1/2$
- Why Gaussian model? Flexible multivariate distribution that is easy to specify and estimate!

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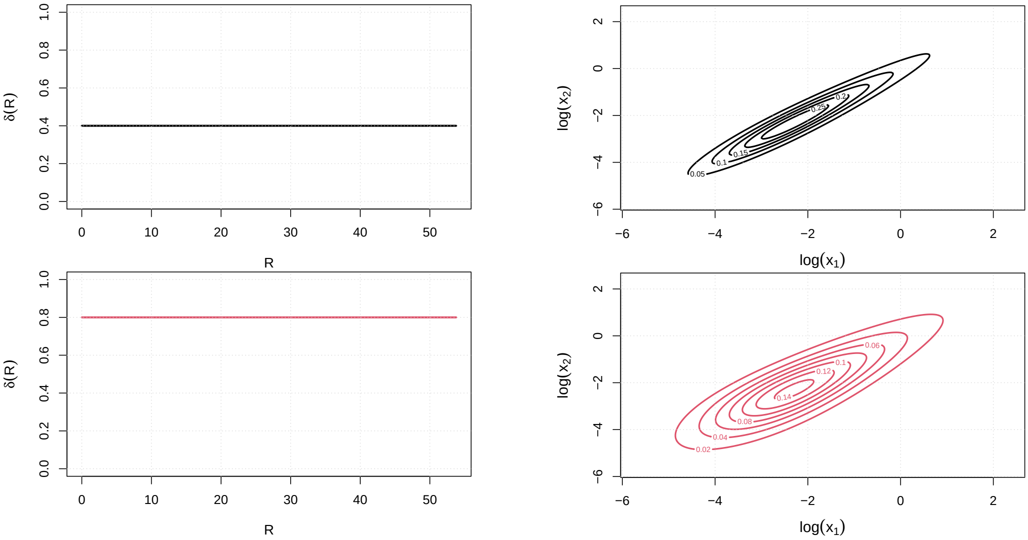
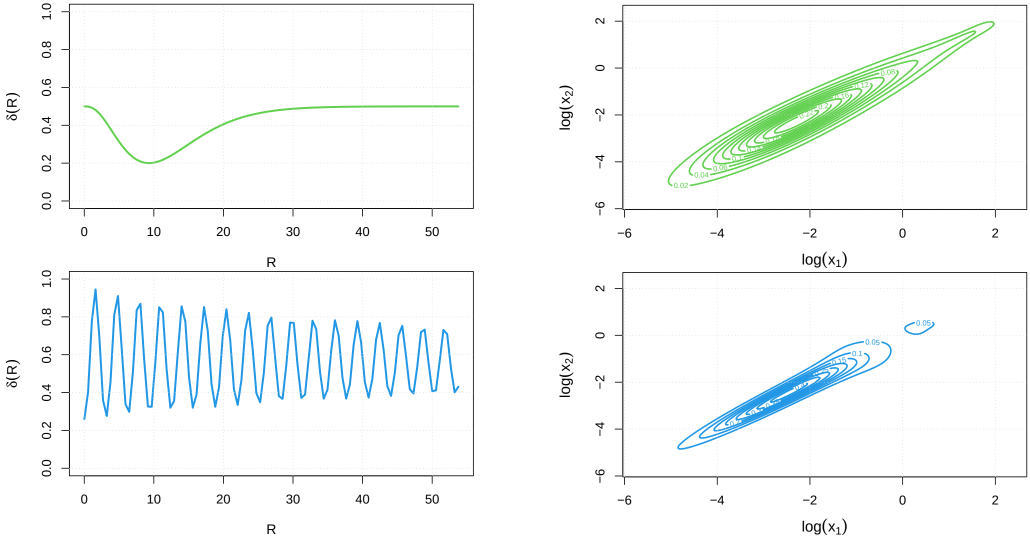
FLEXIBILITY (I)



FLEXIBILITY (II)



Does this work in practice?

ESTIMATION IN TWO STEPS: FIRST STEP

Transform $\boldsymbol{x}_i = (x_{i,1}, x_{i,2})^T$ into $r_i = x_{i,1} + x_{i,2}$

1. Maximize the EGPD log-likelihood

$$l_R(\kappa, \xi) = \sum_{i=1}^{n} \left\{ \log \kappa + (\kappa - 1) \log H_\xi(r_i) + \log h_\xi(r_i) + \log \widehat{b}(H_\xi(r_i)^\kappa) \right\}.$$

- Density $b(u)$ is approximated with **Bernstein polynomials**

$$\widehat{b}(u) = m \sum_{k=1}^{m} \omega_{k,m} \beta_{k,m-k+1}(u)$$

  with $\beta_{i,j}(u) = \binom{j}{i} u^i (1-u)^{j-i}$ and $\omega_{k,m} = \mathbb{B}_n(k/m) - \mathbb{B}_n((k-1)/m)$ [$\mathbb{B}_n$ ECDF of $b_i = H_\xi(\|\boldsymbol{x}_i\|)^\kappa$]

- **Ad-hoc R program but simple to write!**

```
evd::dgpd(x = x,loc = 0,scale = 1,shape = xi)
evd::pgpd(x = x,loc = 0,scale = 1,shape = xi)
ecdf(b)
dbeta(u,shape1 = i, shape2 = j-i+1)
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ESTIMATION IN TWO STEPS: SECOND STEP

Transform $\boldsymbol{x}_i = (x_{i,1}, x_{i,2})^T$ into $v_i = \log(x_{i,1}) - \log(x_{i,2})$

2. Maximize the penalized Gaussian log-likelihood

$$PL_V(\gamma) = -\sum_{i=1}^{n} \left\{ \log(\delta(r_i)) + 0.5 \left(\frac{v_i}{\delta(r_i)} \right)^2 \right\} + \lambda \boldsymbol{\gamma}^\top \boldsymbol{P} \boldsymbol{\gamma}.$$

   * Linear combination of K basis functions $S_j(r)$ (cubic splines)

$$\log \delta(r) = \gamma_0 + \sum_{j=1}^{K} \gamma_j S_j(r), \qquad \boldsymbol{\gamma} = (\gamma_0, \ldots, \gamma_K)^\top$$

   * $\lambda > 0$ smoothing parameter and $\boldsymbol{P}$ is a positive semi-definite matrix
   * **R code**

```
mgcv::gam(list(v~1,~s(r),method = "REML",family=gaulss())
```

ESTIMATION IN TWO STEPS: SECOND STEP

Transform $\boldsymbol{x}_i = (x_{i,1}, x_{i,2})^T$ into $v_i = \log(x_{i,1}) - \log(x_{i,2})$

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$$PL_V(\gamma) = -\sum_{i=1}^{n} \left\{ \log(\delta(r_i)) + 0.5 \left(\frac{v_i}{\delta(r_i)} \right)^2 \right\} + \lambda \boldsymbol{\gamma}^\top \boldsymbol{P} \boldsymbol{\gamma}.$$

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$$\log \delta(r) = \gamma_0 + \sum_{j=1}^{K} \gamma_j S_j(r), \qquad \gamma = (\gamma_0, \ldots, \gamma_K)^\top$$

* $\lambda > 0$ smoothing parameter and $\boldsymbol{P}$ is a positive semi-definite matrix
* **R code**

```
mgcv::gam(list(v~1,~s(r),method = "REML",family=gaulss())
```

ESTIMATION IN TWO STEPS: SECOND STEP

Transform $\boldsymbol{x}_i = (x_{i,1}, x_{i,2})^T$ into $v_i = \log(x_{i,1}) - \log(x_{i,2})$

2. Maximize the penalized Gaussian log-likelihood

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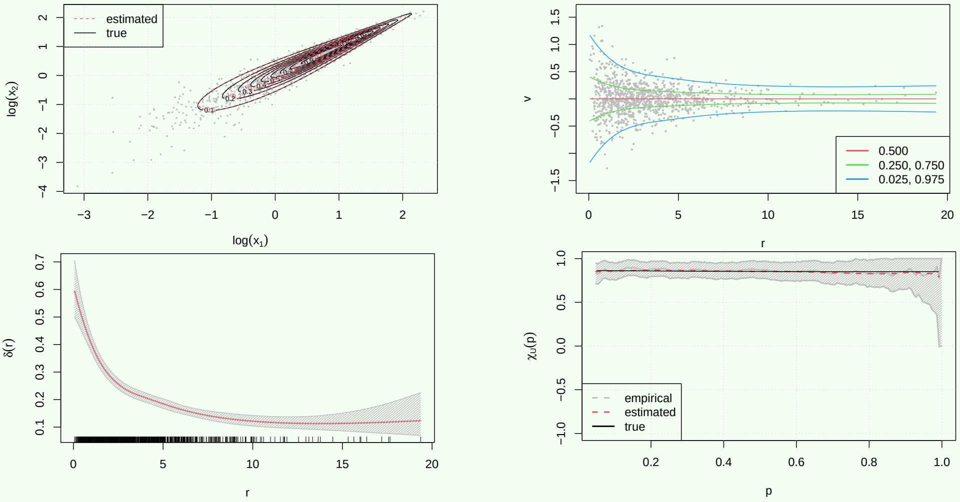
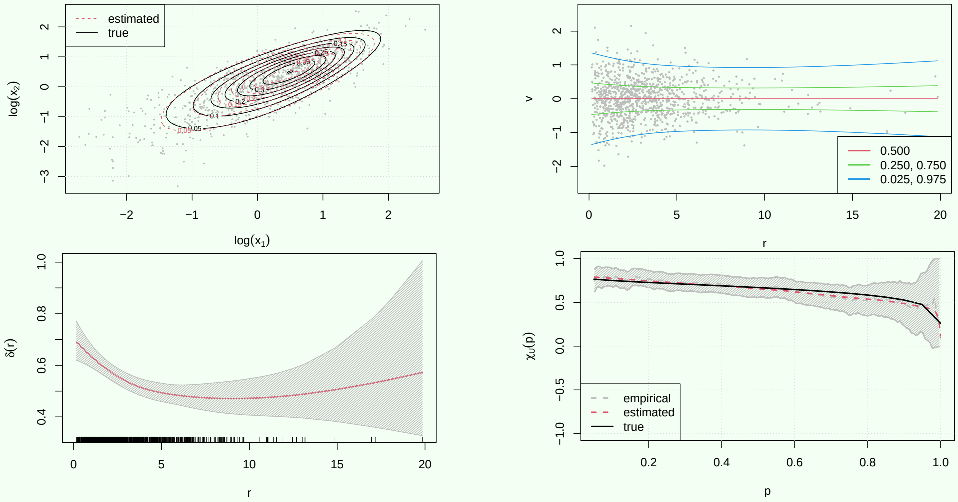
```
mgcv::gam(list(v~1,~s(r),method = "REML",family=gaulss())
```

Can we approximate common (bivariate) copula models ?

MRV: SYMMETRIC LOGISTIC COPULA

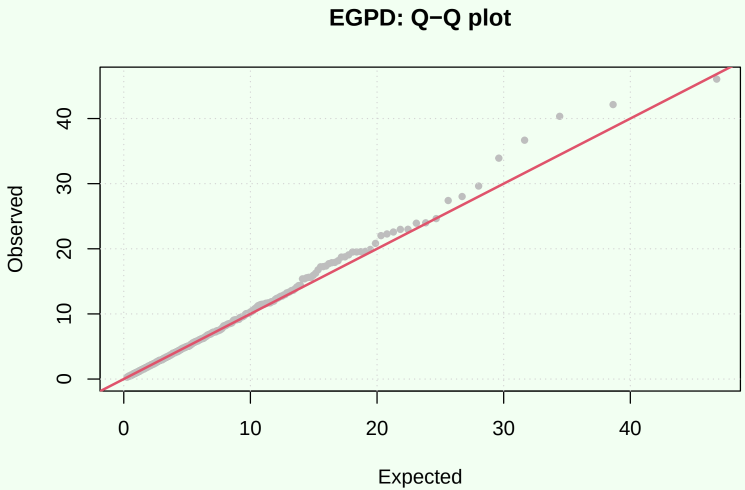


NO MRV: GAUSSIAN COPULA



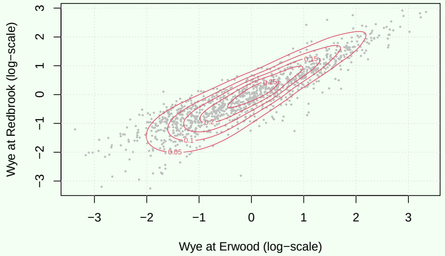
Coming back to the real data example ...

RIVER DISCHARGES: DISTRIBUTION OF $\|\boldsymbol{X}\| = X_1 + X_2$

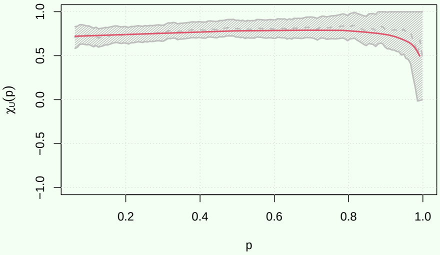


RIVER DISCHARGES: DISTRIBUTION OF $\boldsymbol{X} = (X_1, X_2)^T$

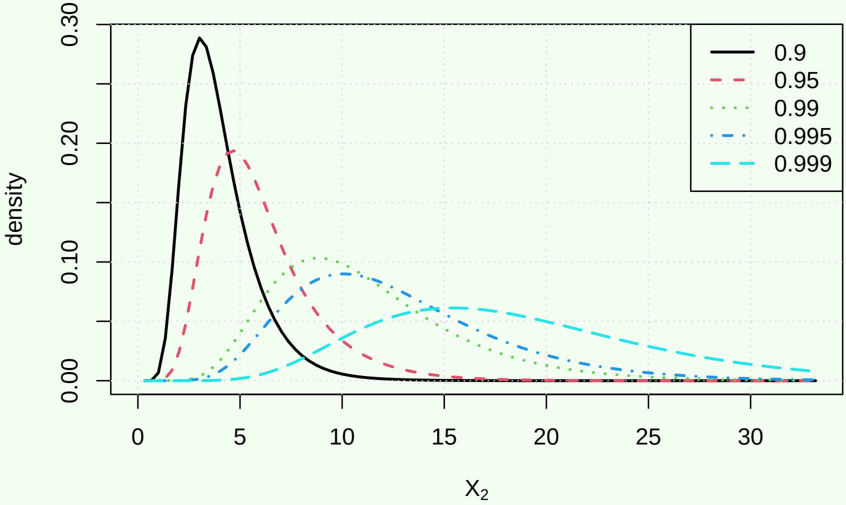
Bivariate density



Goodness of fit



RIVER DISCHARGES: DISTRIBUTION OF $X_2|X_1$



TAKE HOME MESSAGE

A multivariate EGPD exists

RELATED WORKS I

- Ailliot, P., **Gaetan C. and Naveau, P.**, (2025+) A parsimonious tail compliant multiscale statistical model for aggregated rainfall, *submitted*.
- $X_1 + \ldots + X_d \sim EGPD(\kappa, \xi, B_d)$
- Duration d
- Intensity Duration Frequency (IDF) curve
- Rainfall measurement are discrete ...
- ...and most of the time, it is not raining, i.e. zero inflation !

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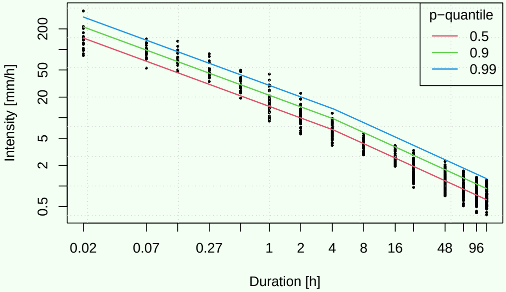

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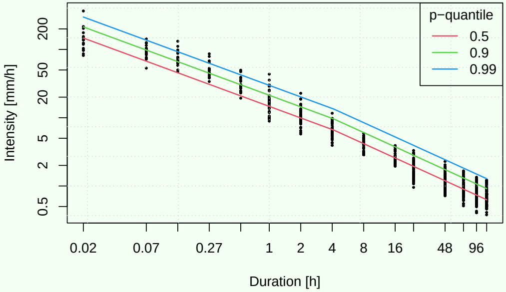

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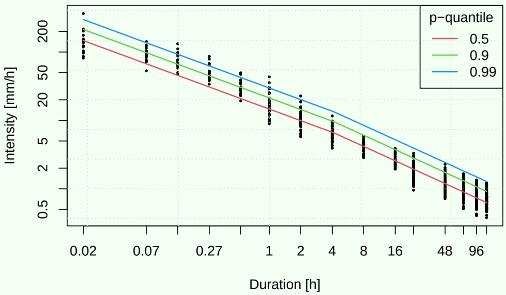

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Statistics > Methodology

[Submitted on 7 Sep 2025]

Joint modeling of low and high extremes using a multivariate extended generalized Pareto distribution

Noura Alotaibi, Matthew Sainsbury-Dale, Philippe Naveau, Carlo Gaetan, Raphaël Huser

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Thanks !!!

Merci !!!

Let $\boldsymbol{X} = \|\mathbf{X}\| \times \boldsymbol{U}$ be a random vector that satisfies (1) and (2).
We say that $\mathbf{X}$ follows a **multivariate logistic-heteroscedastic EGPD** if the log-ratio of its angular component $\boldsymbol{U}$ can be expressed, given the radius $\|\mathbf{X}\| = r$, as

$$V_i := \log(U_i/U_d) = \delta(r)\, Z_i, \ \text{ for } i = 1, \ldots, (d-1), \tag{3}$$

where the (d-1) dimensional vector $\boldsymbol{Z} = (Z_1, \ldots, Z_{d-1})^\top$ is a **zero-mean exchangeable random vector** independent of $\|\mathbf{X}\|$ and $\delta(\cdot)$ is a positive measurable function such that, uniformly on any compact of the real line,

$$\lim_{r \to 0^+} \delta(r) = \delta_- \text{ and } \lim_{r \to \infty} \delta(r) = \delta_+, \tag{4}$$

for some finite positive constants δ_- and δ_+.